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SUPERMODULAR COLOURINGS

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ABSTRACT. We investigate analogies between matroids and certain colourings, or partitions, derived from supermodular functions. We describe a greedy algorithm for minimum colourings, and discuss an intersection theorem.

1. Introduction

A collection C of subsets of a finite set S is called an *intersecting family* if C satisfies:

(1) if $T, U \in C$ and $T \cap U \neq \emptyset$, then $T \cap U \in C$ and $T \cup U \in C$.

A function $g: C \to \mathbb{R}$ is called *supermodular* (*on intersecting pairs*) if:

(2) $g(T \cap U) + g(T \cup U) \geq g(T) + g(U)$ for $T, U \in C$ with $T \cap U \neq \emptyset$.

It is well-known from the results of Edmonds [1] that if $g: C \to \mathbb{Z}$ is a supermodular function on the intersecting family C satisfying

(3) $g(T) \leq |T|$ for all T in C,

then the collection

(4) $\mathcal{S}_g := \{U \subseteq S \mid |T \cap U| \geq g(T) \text{ for all } T \text{ in } \mathcal{C}\}$

is the collection of spanning sets of a matroid on S. With the greedy algorithm one can find a set of minimum cardinality in $\mathcal{S}_g$. This algorithm also shows that

(5) $$\min\{|U| \mid U \in \mathcal{S}_g\} = \max\{g(T_1)+\ldots+g(T_k) \mid T_1,\ldots,T_k \text{ are pairwise disjoint sets in } \mathcal{C} \ (k \geq 0)\}.$$

Similarly, the greedy algorithm gives a minimum weighted spanning set, and a min-max relation for this minimum weight.

Moreover, if $g_1:\mathcal{C}_1 \to \mathbb{Z}$ and $g_2:\mathcal{C}_2 \to \mathbb{Z}$ are supermodular functions on the intersecting families $\mathcal{C}_1$ and $\mathcal{C}_2$ on S, both satisfying (3), then

(6) $$\min\{|U| \mid U \in \mathcal{S}_{g_1} \cap \mathcal{S}_{g_2}\} = \max\{g_1(T_1)+\ldots+g_1(T_k)+ +g_2(V_1)+\ldots+g_2(V_\ell) \mid T_1,\ldots,T_k \in \mathcal{C}_1;\ V_1,\ldots,V_\ell \in \mathcal{C}_2; T_1,\ldots,T_k,V_1,\ldots,V_\ell \text{ pairwise disjoint}\}.$$

Instead of matroids, in this paper we discuss similar results for a "polar" type of combinatorial objects, in terms of colourings related to supermodular functions. In Section 2 we describe a greedy algorithm finding minimum colourings, and in Section 3 we discuss an intersection theorem for colourings. The latter theorem is used in [8] to prove the following result:

(7) *Let* $\mathcal{C}$ *be a crossing family of subsets of the finite set* V *(i.e., if* $T,U \in \mathcal{C}$ *and* $T \cap U \neq \emptyset$, $T \cup U \neq V$ *then* $T \cap U, T \cup U \in \mathcal{C}$*) with* $\emptyset, V \notin \mathcal{C}$*; then the following are equivalent:*

(i) for each directed graph $D=(V,A)$ *the minimum size*

of a cut $\delta^-_A(T)$ (*:= the set of arcs in* A *entering* T) *for* T *in* $\mathcal{C}$, *is equal to the maximum number* k *of pairwise disjoint subsets* $A_1,\ldots,A_k$ *of* A *such that each* T *in* $\mathcal{C}$ *is entered by at least one arc in each of the* A_i;

(*ii*) *there are no* V_1,V_2,V_3,V_4,V_5 *in* $\mathcal{C}$ *such that* $V_1 \subseteq V_2 \cap V_3$, $V_2 \cup V_3 = V$, $V_3 \cup V_4 \subseteq V_5$, $V_3 \cap V_4 = \emptyset$.

2. A greedy algorithm

Let $g:\mathcal{C} \to \mathbb{Z}$ be a supermodular function on the intersecting family $\mathcal{C}$ on S, satisfying (3). Consider the collection

(8) Π_g:=the collection of all collections $\mathcal{F} = \{U_1,\ldots,U_k\}$ of pairwise disjoint subsets of S such that each set T in $\mathcal{C}$ intersects at least g(T) of the U_i.

From (3) it follows that Π_g is non-empty, as $\{\{s\} \mid s \in S\}$ belongs to Π_g. Clearly, if $\mathcal{F} \in \Pi_g$, then

(9) $$|\mathcal{F}| \geq \max_{T \in \mathcal{C}} g(T) .$$

We show that the following greedy algorithm will find a collection $\mathcal{F}$ in Π_g achieving equality in (9), implying that it has minimum cardinality. In this greedy algorithm we assume that for any collection of pairwise disjoint subsets of S we can determine, in polynomial time, whether the collection belongs to Π_g. This is in line with a similar assumption for the greedy algorithm for matroids — see Remark 1 below.

Greedy algorithm for colourings. Order $S = \{s_1,\dots,s_n\}$ arbitrary. Apply the following m-th *iteration*, for $m=1,\dots,n$. Suppose we have found pairwise disjoint non-empty subsets $U_1,\dots,U_k$ of $\{s_1,\dots,s_{m-1}\}$ such that $\{U_1,\dots,U_k,\{s_m\},\dots,\{s_n\}\}$ belongs to Π_g. (If $m=0$ then $k=0$.)

(10) (i) If $\{U_1,\dots,U_k,\{s_{m+1}\},\dots,\{s_n\}\}$ is in Π_g, do not reset;

(ii) Otherwise, if $\{U_1,\dots,U_{i-1},U_i \cup \{s_m\},U_{i+1},\dots,U_k, \{s_{m+1}\},\dots,\{s_n\}\}$ is in Π_g for a certain i, reset $U_i := U_i \cup \{s_m\}$;

(iii) Otherwise, let $U_{k+1} := \{s_m\}$ and reset $k:=k+1$.

At the end of the n-th iteration, let $F := \{U_1,\dots,U_k\}$. Then clearly $F \in \Pi_g$. We show that this collection has equality in (9), and hence is of minimum cardinality.

We use the following notation: if $X_1,\dots,X_n,X$ are sets, then

(11) $$h_{X_1,\dots,X_t}(X) = \text{the number of } i=1,\dots,t \text{ with } X_i \cap X \neq \emptyset.$$

Then for each fixed $X_1,\dots,X_t$, the function $h_{X_1,\dots,X_t}$ is a submodular function. Note that if f is a submodular and g is a supermodular function on the intersecting family C, such that $f(T) \geq g(T)$ for all T in C, then the collection of all sets T in C with $f(T) = g(T)$ is an intersecting family again.

THEOREM 1. *The greedy algorithm described above finds a collection* F *in* Π_g *of minimum cardinality, with* $|F| = \max_{T \in C} g(T)$ *(assuming this maximum is nonnegative).*

PROOF. Let the above algorithm give a collection $\mathcal{F}=\{U_1,\ldots,U_k\}$ in Π_g with $|\mathcal{F}|=k$, and suppose that, in the m-th iteration, s_m was chosen as the first element of the k-th set U_k. So for $i=1,\ldots,k-1$, the collection

$$(12) \qquad \{U_1,\ldots,U_{i-1},U_i\cup\{s_m\},U_{i+1},\ldots,U_{k-1},\{s_{m+1}\},\ldots,\{s_n\}\}$$

does not belong to Π_g. Hence by definition of Π_g, for $i=1,\ldots,k-1$, there exists a set T_i in $\mathcal{C}$ such that

$$(13) \quad h_{U_1,\ldots,U_{i-1},U_i\cup\{s_m\},U_{i+1},\ldots,U_{k-1},\{s_{m+1}\},\ldots,\{s_n\}}(T_i)<g(T_i).$$

Since on the other hand for each i,

$$(14) \qquad h_{U_1,\ldots,U_{k-1},\{s_m\},\{s_{m+1}\},\ldots,\{s_n\}}(T_i)\geqq g(T_i) \;,$$

one easily shows that $s_m\in T_i$, $U_i\cap T_i\neq\emptyset$, and that one has equality in (14). Since the left hand side in (14) is submodular, equality in (14) is closed under taking intersections and unions of the T_i, and hence

$$(15) \quad h_{U_1,\ldots,U_{k-1},\{s_m\},\{s_{m+1}\},\ldots,\{s_n\}}(T_1\cup\ldots\cup T_{k-1})=g(T_1\cup\ldots\cup T_{k-1}).$$

Since the left hand side of (15) is at least k, we know that

$g(T_1 \cup \ldots \cup T_{k-1}) \geq k$, and hence $|F| \leq \max_{T \in C} g(T)$. The converse inequality being trivial, we have proved the theorem. •

REMARK 1. In the greedy algorithm we assumed that any collection of pairwise disjoint subsets of S can be tested to be in Π_g. This is in line with the greedy algorithm for finding a minimum-sized spanning set in a matroid: there we need to be able to test whether a given subset is spanning or not. If the supermodular function is given by an oracle, and the spanning sets are as in (4), then there is a polynomial-time algorithm for testing a set to be spanning, based on the ellipsoid method, but as yet no direct "combinatorial" method has been found. Similarly, for any collection $F=\{U_1,\ldots,U_k\}$ of pairwise disjoint subsets of S one can test in polynomial time whether F belongs to Π_g, by determining

$$(16) \qquad \min_{T \in C} (h_{U_1,\ldots,U_k}(T) - g(T)) ,$$

which is the minimum of a submodular function, and can hence be determined in polynomial time with the ellipsoid method - see [4]. F belongs to Π_g if and only if the minimum (16) is nonnegative.

The above greedy algorithm in fact gives an optimal collection in Π_g also for a certain weighted problem. If $w:S \to \mathbb{R}$, we can find a collection F in Π_g which minimizes

(17) $$\Sigma_{U\in F} \max_{u\in U} w(u) \quad .$$

To this end one should use the ordering $s_1,\ldots,s_n$ of the elements of S with $w(s_1)\geq\ldots\geq w(s_n)$, analogous to the greedy algorithm for minimum weighted spanning sets in matroids.

3. An intersection theorem

A further analogy between spanning sets in matroids and supermodular colourings is provided by the following intersection theorem for supermodular functions.

THEOREM 2. *Let* $g_1:C_1\to\mathbb{Z}$ *and* $g_2:C_2\to\mathbb{Z}$ *be supermodular functions on the intersecting families* C_1 *and* C_2 *on the finite set* S, *such that* $g_j(T)\leq|T|$ *for* $j=1,2$ *and* $T\in C_j$. *Then the minimum size of a collection in* $\Pi_{g_1}\cap\Pi_{g_2}$ *is equal to* $\max\{g_j(T)\mid j=1,2;\ T\in C_j\}$ *(provided that this maximum is nonnegative).*

PROOF. Clearly, the maximum does not exceed the minimum. To prove the converse, we use the submodular function defined in (11).

Let $k:=\max\{g_j(T)\mid j=1,2;\ T\in C_j\}$. The theorem being trivial if $k=0$, we may assume $k\geq 1$. Let $U_1,\ldots,U_k$ be pairwise disjoint subsets of S such that:

(18) $$g_j(T)\leq h_{U_1,\ldots,U_k}(T) + |T\setminus(U_1\cup\ldots\cup U_k)|$$

for $j=1,2$ and $T\in C_j$, and such that

(19) $$|U_1 \cup \ldots \cup U_k| \text{ is as large as possible.}$$

Such $U_1,\ldots,U_k$ exist, as $U_1=\ldots=U_k=\emptyset$ satisfies (18). We are finished when we have shown that $U_1 \cup \ldots \cup U_k = S$, since then $\{U_1,\ldots,U_k\} \in \Pi_{g_1} \cap \Pi_{g_2}$. Suppose to the contrary there is an s in $S \setminus (U_1 \cup \ldots \cup U_k)$.

Then there will exist an i_1 such that if we replace U_{i_1} by $U_{i_1} \cup \{s\}$, then (18) is still satisfied for $j=1$. Otherwise, for all $i=1,\ldots,k$, there would exist a set T_i in C_1 such that

(20) $$g_1(T_i)>h_{U_1,\ldots,U_{i-1}\cup s,U_{i+1},\ldots,U_k}(T_i)+|T_i \setminus (U_1 \cup \ldots \cup U_k \cup s)|.$$

Combined with (18) for the original $U_1,\ldots,U_k$, this implies that T_i contains s and $T_i \cap U_i \neq \emptyset$, and that (18) holds with equality for $j=1$ and $T=T_i$. Now the collection of sets T satisfying (18) with equality is an intersecting family (as the left hand side is supermodular and the right hand side is submodular). Hence the union $T_0:=T_1 \cup \ldots \cup T_k$ satisfies (18) with equality. But then

(21) $$g_1(T_0) = h_{U_1,\ldots,U_k}(T_0) + |T_0 \setminus (U_1 \cup \ldots \cup U_k)| \geq k+1$$

(as T_0 contains s and intersects all U_i). (21) contradicts the definition of k.

Similarly, there exists an i_2 such that if we replace U_{i_2} by $U_{i_2} \cup \{s\}$, then (18) is still satisfied for $j=2$.

Now, $i_1 \neq i_2$, since otherwise we could replace U_{i_1} by $U_{i_1} \cup \{s\}$, without violating (18) for $j=1,2$, contradicting (19).

We may assume that $i_1 = 1$ and $i_2 = 2$. Now for $j=1,2$ and $T \in C_j$ one has:

$$(22) \qquad g_j(T) \le h_{U_3,\dots,U_k}(T) + |T \setminus (U_1 \cup \dots \cup U_k \cup s)| + 2 .$$

For $j=1$ this follows from the fact that we could augment U_1 with s:

$$(23) \qquad \begin{aligned} g_1(T) &\le h_{U_1\cup s,U_2,\dots,U_k}(T) + |T \setminus (U_1 \cup s \cup U_2 \cup \dots \cup U_k)| = \\ &= h_{U_3,\dots,U_k}(T) + |T \setminus (U_1 \cup \dots \cup U_k \cup s)| + h_{U_1\cup s,U_2}(T) \le \\ &\le h_{U_3,\dots,U_k}(T) + |T \setminus (U_1 \cup \dots \cup U_k \cup s)| + 2 . \end{aligned}$$

For $j=2$ (23) is shown similarly.

Let $V_1,\dots,V_m$ be the minimal sets T in C_1 satisfying (22) for $j=1$ with equality (minimal with respect to inclusion). As the collection of sets T in C_1 satisfying (22) with equality (for $j=1$) is an intersecting family, the sets $V_1,\dots,V_m$ are pairwise disjoint. Moreover, as equality in (22) implies equality throughout in (23), we know that $h_{U_1\cup s,U_2}(V_i) = 2$, and hence that

$|V_i \cap (U_1 \cup U_2 \cup s)| \geq 2$ for $i=1,\ldots,m$.

Similarly, let $W_1,\ldots,W_n$ be the minimal sets in C_2 which satisfy (22) with equality for $j=2$. Again, $W_1,\ldots,W_n$ are pairwise disjoint, and $W_i \cap (U_1 \cup U_2 \cup s)| \geq 2$ for $i=1,\ldots,n$.

Now $U_1 \cup U_2 \cup s$ can be split into classes U_1' and U_2' such that both U_1' and U_2' intersect each of the sets $V_1,\ldots,V_m,W_1,\ldots,W_n$. To see this, choose pairs $e_1,\ldots,e_m,f_1,\ldots,f_n$ as subsets of $U_1 \cup U_2 \cup s$ such that $e_1 \subseteq V_1,\ldots,e_m \subseteq V_m, f_1 \subseteq W_1,\ldots$ $\ldots,f_n \subseteq W_n$. Since $e_1,\ldots,e_m$ are pairwise disjoint, and since $f_1,\ldots,f_n$ are pairwise disjoint, it follows that the edges $e_1,\ldots,e_m,f_1,\ldots,f_n$ make up a bipartite graph, with vertex set $U_1 \cup U_2 \cup s$. Then any two-colouring of this bipartite graph gives a splitting into classes U_1' and U_2' as required.

We finally show that replacing U_1 and U_2 by U_1' and U_2' does not violate (18) for $j=1,2$, which however contradicts the maximality of $|U_1 \cup \ldots \cup U_k|$.

So we have to prove:

$$(24) \qquad g_j(T) \leq h_{U_1',U_2',U_3,\ldots,U_k}(T) + |T \setminus (U_1' \cup U_2' \cup U_3 \cup \ldots \cup U_k)|$$

for $j=1,2$ and $T \in C_j$. First let $j=1$, and choose $T \in C_1$. If T includes one of the V_i as a subset, then T intersects both U_1' and U_2' (as V_i intersects both of these sets). In this case, by (22),

$$(25) \qquad \begin{aligned} g_1(T) &\le h_{U_3,\ldots,U_k}(T) + |T \setminus (U_1 \cup \ldots \cup U_k \cup s)| + 2 = \\ &= h_{U_1',U_2',U_3,\ldots,U_k}(T) + |T \setminus (U_1' \cup U_2' \cup U_3 \cup \ldots \cup U_k)|. \end{aligned}$$

If T includes none of the V_i, then the inequality (22) for $j=1$ is strict (by definition of $V_1,\ldots,V_m$). So if T intersects $U_1' \cup U_2'$ then

$$(26) \qquad \begin{aligned} g_1(T) &\le h_{U_3,\ldots,U_k}(T) + |T \setminus (U_1 \cup \ldots \cup U_k \cup s)| + 1 \le \\ &\le h_{U_1',U_2',U_3,\ldots,U_k}(T) + |T \setminus (U_1' \cup U_2' \cup U_3 \cup \ldots \cup U_k)|. \end{aligned}$$

If T does not intersect $U_1' \cup U_2'$, then

$$(27) \qquad \begin{aligned} g_1(T) &\le h_{U_1,\ldots,U_k}(T) + |T \setminus (U_1 \cup \ldots \cup U_k)| = \\ &= h_{U_1',U_2',U_3,\ldots,U_k}(T) + |T \setminus (U_1' \cup U_2' \cup U_3 \cup \ldots \cup U_k)|. \end{aligned}$$

The inequality (24) for $j=2$ is shown similarly. •

The proof also shows that a collection in $\Pi_{g_1} \cap \Pi_{g_2}$ of minimum size can be found in polynomial time, by minimizing certain submodular functions, which can be done in polynomial time with the ellipsoid method (cf. [4]). We do not know a min-max relation or a polynomial algorithm for finding a minimum-weighted collection in $\Pi_{g_1} \cap \Pi_{g_2}$ (with respect to the weight function (17)).

REMARK 2. Theorem 2 can be formulated in terms of generalized polymatroids (cf. Frank [3]). If $g:\mathcal{C} \to \mathbb{R}$ is a supermodular function on the intersecting family of subsets of S, let the polyhedron P_g in $\mathbb{R}_+^S$ be defined by:

$$(28) \qquad P_g := \{x \in \mathbb{R}_+^S \mid x(T) \geq g(T) \quad \text{for} \quad T \in \mathcal{C}\},$$

where $x(T) := \Sigma_{s \in T}\, x(s)$. It is known (cf. [1],[3]) that if g is integer-valued, the polyhedron P_g is integral (i.e., each vertex of P_g is integral). Now Theorem 2 implies the following. Let $g_1:\mathcal{C}_1 \to \mathbb{Z}$ and $g_2:\mathcal{C}_2 \to \mathbb{Z}$ be supermodular functions on the intersecting families $\mathcal{C}_1$ and $\mathcal{C}_2$ on S, and let $k := \{\max g_j(T) \mid j=1,2;\ T \in \mathcal{C}_j\}$. Then if b is an integral vector in $P_{g_1} \cap P_{g_2}$, there are nonnegative integral vectors $b_1,\ldots,b_k$ such that

$$(29) \qquad \begin{array}{ll} \text{(i)} & b = b_1 + \ldots + b_k \ ; \\ \text{(ii)} & \text{for } j=1,2 \text{ and } T \in \mathcal{C}_j : \displaystyle\sum_{i=1}^{k} \min\{b_i(T), 1\} \geq g_j(T) \end{array}$$

This follows from Theorem 2 by splitting each element s of S into $b(s)$ copies.

We conclude with mentioning some applications of Theorem 2.

APPLICATION 1. Let $G = (V,E)$ be a bipartite graph, with colour classes V_1 and V_2, and let for $j=1,2$:

(30) $$\mathcal{C}_j := \{\delta(v) \mid v \in V_j\}$$

where $\delta(v)$ denotes the set of edges incident with vertex v. Clearly, $\mathcal{C}_1$ and $\mathcal{C}_2$ are intersecting families. If we define $g_j(\delta(v)) = |\delta(v)|$ for $j=1,2$ and $v \in V_j$, we obtain supermodular functions g_1 and g_2 on $\mathcal{C}_1$ and $\mathcal{C}_2$, satisfying (3). Theorem 2 now gives König's edge-colouring theorem [6]: the edge-colouring number of G is equal to the maximum degree of G. If we define $g_j(\delta(v)) = k$ for $j=1,2$ and $v \in V_j$, where k is the minimum degree of G, Theorem 2 gives a result of Gupta [5]: the maximum number of pairwise disjoint edge sets in G, each covering all vertices, is equal to the minimum degree of G. If we define $g_j(\delta(v)) = \min\{k, |\delta(v)|\}$, for $j=1,2$ and $\delta(v) \in \mathcal{C}_j$, where k is an arbitrary natural number, Theorem 2 gives a result of De Werra [9].

APPLICATION 2. We will indicate how to derive from Theorem 2 the following "disjoint bi-branching theorem" ([7]):

(31) *Let* $D = (V,A)$ *be a directed graph, and let* V *be split into classes* V_1 *and* V_2. *Suppose that each* $V' \subseteq V$ *with* $\emptyset \neq V' \subseteq V_1$ *or* $V_1 \subseteq V' \neq V$ *is entered by at least* k *arcs of* D. *Then* A *can be split into classes* $A_1, \ldots, A_k$ *such that for each* $i=1,\ldots,k$ *and for each* $v \in V_1$ *there is a directed path in* A_i *from* V_2 *to* v, *and for each* $v \in V_2$ *there is a directed path in* A_i *from* v *to* V_1.

This result is one of the auxiliary theorems for the min-max relation proved in [7], which is the special case of (7) where $\mathcal{C} \cup \{\emptyset, V\}$ is closed under taking any union and intersection.

To prove (31), Theorem 2 is combined with the following result of Edmonds [2], using the notation (11) and $d_A^-(V') :=$ the number of arcs in A entering V' :

(32) *if* $D = (V,A)$ *is a directed graph, and* $R_1,\ldots,R_k$ *are subsets of* V *such that*

$$d_A^-(V') + h_{R_1,\ldots,R_k}(V') \geq k$$

for each nonempty subset V' *of* V, *then* A *can be split into classes* $A_1,\ldots,A_k$ *such that for each* $i=1,\ldots,k$ *and each* $v \in V \setminus R_i$, *there is a directed path in* A_i *starting in* R_i *and ending in* v.

Taking $R_1 = \ldots = R_k = \{r\}$ gives Edmonds' disjoint branching theorem. (31) can be seen as a result on "glueing branchings together to obtain bi-branchings". Let

(33) $A^\circ := \{a \in A \mid a$ has tail in V_2 and head in $V_1\}$,

$A' := \{a \in A \mid a$ has both tail and head in $V_1\}$;

$A'' := \{a \in A \mid a$ has both tail and head in $V_2\}$.

Let furthermore,

$$(34)\quad \mathcal{C}_1 := \{\delta^-_{A^\circ}(V') \mid \emptyset \neq V' \subseteq V_1\} ,$$
$$\mathcal{C}_2 := \{\delta^-_{A^\circ}(V') \mid V_1 \subseteq V' \neq V\}$$

Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are intersecting families on A°. Define for $j=1,2$, $g_j : \mathcal{C}_j \to \mathbb{Z}$ by

$$(35)\quad g_1(B) := \max\{k-d^-_{A'}(V') \mid \emptyset \neq V' \subseteq V_1 ,\ \delta^-_{A^\circ} = B\} \qquad \text{for } B \in \mathcal{C}_1 ,$$
$$g_2(B) := \max\{k-d^-_{A''}(V') \mid V_1 \subseteq V' \neq V ,\ \delta^-_{A^\circ} = B\} \qquad \text{for } B \in \mathcal{C}_2 .$$

Then g_1 and g_2 are supermodular on intersecting pairs. Moreover, if V' attains the maximum in (35) then

$$(36)\quad g_1(B) = k - d^-_{A'}(V') \leq d^-_A(V') - d^-_{A'}(V') = d^-_{A^\circ}(V') = |B| ,$$
$$g_2(B) = k - \bar{d}_{A''}(V') \leq d^-_A(V') - d^-_{A''}(V') = d^-_{A^\circ}(V') = |B|$$

Since $g_j(B) \leq k$ for $j=1,2$ and $B \in \mathcal{C}_j$, we can split, by Theorem 2, A° into classes $A^\circ_1,\ldots,A^\circ_k$ such that:

(37) if $\emptyset \neq V' \subseteq V_1$, V' is entered by at least $k-d^-_{A'}(V')$ of the classes A°_i,
if $V_1 \subseteq V' \neq V$, V' is entered by at least $k-d^-_{A''}(V')$ of the classes A°_i.

We leave it to the reader to combine this result with (32) to obtain (31).

REFERENCES

[1] Edmonds, J., Submodular functions, matroids, and certain polyhedra, in: *Combinatorial Structures and Their Applications* (R. Guy, et al., eds.), Gordon and Breach, New York, 1970, pp. 69-87.

[2] Edmonds, J., Edge-disjoint branchings, in: *Combinatorial Algorithms* (B. Rustin, ed.), Academic Press, New York, 1973, pp. 91-96.

[3] Frank, A., Generalized polymatroids, *Proc. Sixth Hungarian Combinatorial Colloquium* (Eger, 1981), to appear.

[4] Grötschel, M., Lovász, L. and Schrijver, A., The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* 1 (1981) 169-197.

[5] Gupta, R.P., A decomposition theorem for bipartite graphs, in: *Theory of Graphs* (P. Rosenstiehl, ed.), Gordon and Breach, New York, 1967, 137-138.

[6] König, D., Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, *Math. Ann.* 77 (1916) 453-465.

[7] Schrijver, A., Min-max relations for directed graphs, *Annals of Discrete Math.* 16 (1982) 261-280.

[8] Schrijver, A., *Packing and covering of crossing families of cuts*, J. Combinatorial Theory (B), to appear.

[9] de Werra, D., Some remarks on good colorations,
J. Combinatorial Theory (B) 21 (1976) 57-64.

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